in a region $D : \| y(m) \| \leqslant H,\ m = 0, 1, \ldots\ (H = \text{const})$.

Theorem 2. Let the systems (21) and (2) be connected by a relation of the type (5). If the zero solution of the system (2) is exponentially stable, then for sufficiently small γ and M the zero solution of the system (21) will also be exponentially stable.

The proof of Theorem 2 differs from that of Theorem 1 only in the fact that M in the inequality (11) is replaced by $M + \gamma$.

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ON EQUIVALENCE OF THE EQUATIONS OF MOTION OF NONHOLONOMIC SYSTEMS

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We prove the equivalence of the equations of motion of nonholonomic systems with constraints linear in velocities, obtained by various methods. At present, the equations of motion of nonholonomic systems exist in various forms. Naturally, the question of their identity to each other was brought up in [1 — 3], and the problem was also discussed in [4 — 8] and in the dissertation of M. I. Efimov (*).

1. The author of [1 — 3] postulates that the final form of the equations of motion of a system obtained by transforming the general dynamic equations depends on the point at which the equations of nonholonomic constraints are taken into account. He states that in the general case of arbitrary nonholonomic systems with constraints which are linear in velocities, the equations constructed by different methods cannot be guaranteed to be identical. Volterra [9], Appell [10] and MacMillan [11] derive the equations of motion from the general dynamic equation in Cartesian coordinates and bring the nonholonomic constraints into the discussion at once. Hamel [12], Chaplygin [13] and Voronets [14] bring in the nonholonomic constraints after the general dynamic equations have been transformed to the generalized coordinates. In the opinion of the author of [1—3], the equations of motion obtained using the methods of Volterra, Appell and MacMillan on one hand, and the methods of Voronets (Chaplygin) and Hamel on the other hand, will not, in general, be identical, i. e. the systems of equations will not be equivalent to each

*) Efimov, M. I., On the Chaplygin equations for nonholonomic systems. Candidate's dissertation, Inst. mekhaniki, Akad. Nauk SSSR, 1953.

other. To refute this statement, it is sufficient to show that the equations of motion obtained by one of the methods of the first group (Volterra, Appell and MacMillan) coincide with the equations of motion obtained by one on the methods of the second group (Voronets, Hamel). We shall prove that the Appell equations coincide with the Voronets equations for any nonholonomic system with constraints linear in velocities.

Let the nonholonomic constraints imposed on a system be given by the equations

$$q_i^{\cdot} = \sum_{k=1}^{l} b_{ik} q_k^{\cdot} + b_i \quad (i = l+1,\ l+2,\ \ldots,\ n) \tag{1.1}$$

Differentiating the constraint equations with respect to time, we obtain

$$q_i^{\cdot\cdot} = \sum_{k=1}^{l} b_{ik} q_k^{\cdot\cdot} + \cdots \quad (i = l+1,\ l+2, \ldots,\ n) \tag{1.2}$$

in which the terms containing no generalized accelerations have been omitted. The Appell equations for this system have the form [8]

$$\frac{\partial S^*}{\partial q_k^{\cdot\cdot}} = Q_k^* \quad (k = 1,\ 2,\ \ldots,\ l) \tag{1.3}$$

Here S^* denotes the acceleration energy transformed with the nonholonomic constraints taken into account and Q_k^* is the generalized force corresponding to the independent increment δq_k. If S and T denote the acceleration and kinetic energies of the system not yet transformed under the nonholonomic constraints, then [15]

$$\frac{\partial S}{\partial q_\nu^{\cdot\cdot}} \equiv \frac{d}{dt}\frac{\partial T}{\partial q_\nu^{\cdot}} - \frac{\partial T}{\partial q_\nu} \quad (\nu = 1,\ 2,\ \ldots,\ l,\ \ldots,\ n) \tag{1.4}$$

The energy of acceleration S depends on the generalized accelerations $q_k^{\cdot\cdot}$ entering it explicitly, and by means of the relation for $q_i^{\cdot\cdot}$ given by (1.2). Therefore [8]

$$\frac{\partial S^*}{\partial q_k^{\cdot\cdot}} = \frac{\partial S}{\partial q_k^{\cdot\cdot}} + \sum_{i=l+1}^{\nu} \frac{\partial S}{\partial q_i^{\cdot\cdot}} b_{ik} \tag{1.5}$$

Let us denote by T^* the kinetic energy of the system transformed with the constraints (1.1) taken into account. Then

$$\frac{\partial T^*}{\partial q_k^{\cdot}} = \frac{\partial T}{\partial q_k^{\cdot}} + \sum_{i=l+1}^{n} \frac{\partial T}{\partial q_i^{\cdot}} b_{ik} \tag{1.6}$$

Differentiating (1.6) with respect to time, we obtain

$$\left(\frac{d}{dt}\frac{\partial T}{\partial q_k^{\cdot}} + \sum_{i=l+1}^{n} b_{ik}\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}}\right) = \frac{d}{dt}\frac{\partial T^*}{\partial q_k^{\cdot}} - \sum_{i=l+1}^{n}\frac{\partial T}{\partial q_i^{\cdot}}\frac{db_{ik}}{dt} \tag{1.7}$$

Further transforming Eqs. (1.3) using the relations (1.4), (1.5) and (1.7) and eliminating the quantities $\partial T / \partial q_k$ and $\partial T / \partial q_i$ by means of the relations

$$\frac{\partial T^*}{\partial q_\nu} = \frac{\partial T}{\partial q_\nu} + \sum_{i=l+1}^{n} \frac{\partial T}{\partial q_i^{\cdot}}\left(\sum_{j=1}^{l}\frac{\partial b_{ij}}{\partial q_\nu} q_j^{\cdot} + \frac{\partial b_i}{\partial q_\nu}\right) \quad (\nu = 1,\ 2,\ \ldots,\ l,\ \ldots,\ n) \tag{1.8}$$

we obtain the Voronets equations

$$\frac{d}{dt}\frac{\partial T^*}{\partial q_k^{\cdot}} - \frac{\partial T^*}{\partial q_k} - \sum_{i=l+1}^{n} b_{ik}\frac{\partial T^*}{\partial q_i} - \sum_{i=l+1}^{n} B_{ik}\frac{\partial T}{\partial q_i^{\cdot}} = Q_k^* \quad (k = 1,\ 2,\ \ldots,\ l) \tag{1.9}$$

$$B_{ik} \equiv \frac{db_{ik}}{dt} - \sum_{j=1}^{l} \left(\frac{\partial b_{ij}}{\partial q_k} + \sum_{i=l+1}^{n} \frac{\partial b_{ij}}{\partial q_i} b_{ik} \right) q_j^{\cdot} - \frac{\partial b_i}{\partial q_k} - \sum_{i=l+1}^{n} \frac{\partial b_i}{\partial q_i} b_{ik}$$

valid for any nonholonomic system with constraints linear in velocities, since they were derived without any restrictions being placed on the coefficients of the constraint equations and on the kinetic energy of the system.

In the course of transforming the Appell equations into the Voronets equations we did not make any additional physical assumptions, and the only transformations we used were identity transformations. This implies that the Appell and the Voronets equations are identical. We must of course remember that the dependent generalized velocities must be eliminated from the expressions $\partial T / \partial q_i^{\cdot}$ in the Voronets equations, using the constraint equations (1. 1).

In the particular case of Chaplygin systems, the Appell equations coincide with the Chaplygin equations.

A case of a nonholonomic system moving without the influence of the active forces, is given in [3]. The system is determined by three geometrical coordinates q_1, q_2, q_3 with one nonholonomic ideal constraint $q_1 = q_3 q_2^{\cdot}$. The equations of motion are constructed in the Chaplygin and the Appell forms (see formulas (a_9) and (a_{14}) in [3]) and compared, to reach the false conclusion that they differ explicitly from each other. In fact no difference exists; if we use the constraint equation (a_1) to eliminate the dependent generalized velocity $q_1^{\cdot}$ from the expression $\partial T / \partial q_1^{\cdot} \equiv q_1^{\cdot}$ in the Chaplygin equations (a_9), then the equations of motion in the Chaplygin and the Appell forms will become identical.

Naturally, we can also prove the equivalence of the equations of motion of nonholonomic systems constructed by the MacMillan or Volterra methods with those constructed by the Voronets or Hamel methods.

2. Let us now pause on the Hamel equations [12]. In the opinion of the author of [3] the Hamel method can be used to obtain two, quite different systems of equations of motion, depending on at which point we take the nonholonomic constraints into account, whether it is before, or after differentiating the expression for the kinetic energy with respect to the quasi-velocities. But the very process of deriving the Hamel equations shows that these equations contain the derivatives of the kinetic energy T with respect to all quasi-velocities, therefore the nonholonomic constraints must not be taken into account when constructing T; they can be brought in only after calculating the kinetic energy derivatives with respect to the quasi-velocities [8]. This implies that the Hamel method yields a single, unique system of equations of motion, and this system will be equivalent to systems obtained by other methods.

3. We shall point out the error made by the author of [4] in constructing the equations of motion of a gyro gimbal, using the Volterra method. In [4] the author investigated the motion of a gyro gimbal with a linear nonholonomic constraint $\alpha^{\cdot} - \psi^{\cdot} \sin \vartheta = 0$ imposed on it, and the Volterra method was used to derive the system of equations of motion of the gimbal (see Eqs. (20) in [4]). The author used an inconsistent system of five equations with four unknowns. From this system he separated a system of four equations, assumed the fifth equation to be compatible and solved it in the bilinear covariants of the nonholonomic coordinates. This however contradicts the rules of higher algebra. Let us obtain the correct equations of motion of a gyro gimbal in nonholonomic coordinates using the Volterra method given in [15]. Setting $\alpha^\cdot = p_1$, $\beta^\cdot = p_2$, $\gamma^\cdot = p_3$, $\vartheta^\cdot = p_4$, $\psi^\cdot = p_1/\sin\vartheta$ and assuming

$$\delta d\alpha - d\delta\alpha = 0,\ \delta d\beta - d\delta\beta = 0,\ \delta d\gamma - d\delta\gamma = 0,\ \delta d\vartheta - d\delta\vartheta = 0 \tag{3.1}$$

we obtain

$$\delta\psi^\cdot - \frac{d}{dt}\delta\psi = \frac{\cos\vartheta}{\sin^2\vartheta}(\vartheta^\cdot\delta\alpha - \alpha^\cdot\delta\vartheta) \tag{3.2}$$

Further, transforming the general contral equation [8]

$$\frac{d}{dt}\sum_{s=1}^{n}\frac{\partial T}{\partial q_s^\cdot}\delta q_s - \delta T + \sum_{s=1}^{n}\frac{\partial T}{\partial q_s^\cdot}\left(\delta q_s^\cdot - \frac{d}{dt}\delta q_s\right) = \sum_{s=1}^{n} Q_s\delta q_s$$

with the generalized coordinates $q_1 = \alpha$, $q_2 = \beta$, $q_3 = \gamma$, $q_4 = \vartheta$, $q_5 = \psi$ and the relations (3.1) and (3.2) taken into account, we obtain the system of equations of motion of the gyro gimbal. This system coincides with Eqs. (22) of [4] obtained by the Chaplygin method. The equations obtained by the author of [4] using the Hamel method are, as expected, identical to the equations obtained by the Chaplygin method. Thus we find no difference between the equations of motion constructed by different methods for this case, and it follows that the conclusions of the author of [4] were erroneous.

Everything we said above implies that the equations of motion of the nonholonomic systems obtained by different methods will be identical, i.e. their final form does not depend on the stage of computation at which the constraints are taken into account. The choice of the method of constructing the differential equations of motion is governed by the computational convenience, and this point of view was also endorsed by Lur'e in [8].

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ON A VERSION OF THE NONLINEAR DYNAMICAL THEORY OF THIN MULTILAYERED SHELLS

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A version of the geometrically nonlinear theory of elastic multilayered shells subjected to a nonconservative load is proposed. Transverse shear strains in the layers and strains in the direction of the normal to the middle surface are taken into account. As a rule, a description of the nonstationary dynamical processes associated with shell buckling can be performed on the basis of a geometrically nonlinear theory [1]. The behavior of multilayered plates and shells under large deflections has been examined in [2—5]. A variational formulation, which is valid for conservative loads acting on a shell, is used in [5] to derive the geometrically nonlinear equations. The variational principle is formulated in this paper in a form also applicable in the case of no potential of the external forces. One of the advantages of the approach developed here as compared with the results of [5] is the additional possibility of describing the local dynamical buckling of the shell in modes associated with the change in its thickness.

1. Variational principle for a three-dimensional body. The variational principle of elasticity theory for a three-dimensional body under large displacements is written as follows:

$$\delta J_0 = \delta \int_{t_0}^{t_1} \Big(\int_V \Big\{ -\frac{1}{2} E^{ikjl} \varepsilon_{ik} \varepsilon_{jl} + \sigma^{ik} \Big[\varepsilon_{ik} - \frac{1}{2} (\eta_{ik} + \eta_{ki} + \eta_{i\cdot}^{\cdot j} \eta_{kj}) \Big] + \tag{1.1}$$

$$\theta^{ik} (\eta_{ik} - \overline{\nabla}_i u_k) + \frac{1}{2} \rho \frac{\partial u_i}{\partial t} \frac{\partial u^i}{\partial t} \Big\} dV + \int_{S_1} P^i u_i dS +$$

$$\int_{S_2} (u_k - U_k) \theta^{ik} n_i dS \Big) = 0, \quad i, \ k = 1, 2, 3$$

$$\delta P^i = 0 \tag{1.2}$$

Here E^{ikjl} is the elasticity tensor, ρ is the material density, u_i are the displacement